N-S Eqs examples

https://www.youtube.com/watch?v=xLQNqwPUuN4

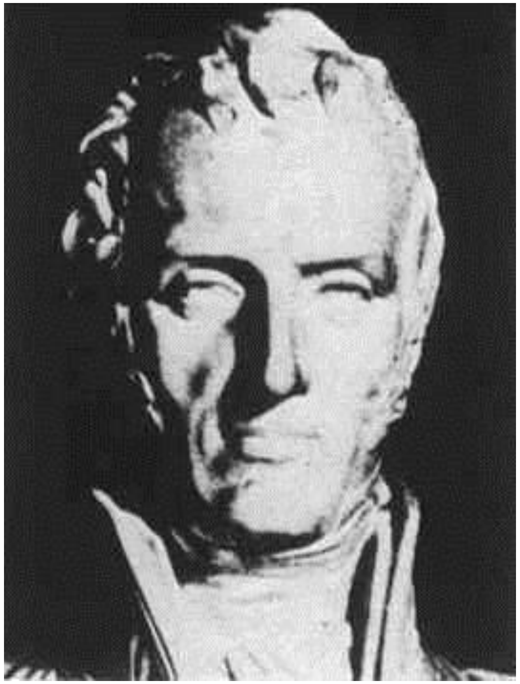
https://www.youtube.com/watch?v=pVLCmT5lkw4&t=17s

Development of Navier-Stokes equation

- Derived by Claude-Louis-Marie **Navier** in **1827**,
- and independently by Siméon-Denis **Poisson** in 1831.
  - Their motivations of the stress tensor were based on what amounts to a molecular view of how stresses are exerted by one fluid particle against another.
- Later, Barré **de Saint Venant** (in 1843)
- George Gabriel **Stokes** (in 1845) derived the equation starting with the **linear** stress rate-of-strain argument.

Development of Navier-Stokes equation

Louis Marie Henri Navier (**1827)**



George Gabriel Stokes (**1845**)

The Development of the Equation

- **Bernoulli** adapted methods of calculus to analyze fluid motion when subjected to various forces.
- **Euler** formulated a set of equations, which combined solutions describe precisely the motion of a viscosity-free fluid.
- **Navier** amended Euler's equations to account for viscosity.
- **Stokes rediscovered** Navier's equations, with proper mathematical reasoning.

N-S eqs

- **Newtonian, constant viscosity and constant density**

$$\rho \frac{D\vec{V}}{Dt} = \rho\vec{g} - \nabla\vec{p} + \mu\nabla^2\vec{V}$$

x-direction:

$$\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}\right)=$$

$$\rho g_x-\frac{\partial p}{\partial x}+\mu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)$$

y-direction:

$$\rho\left(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}\right)=$$

$$\rho g_y-\frac{\partial p}{\partial y}+\mu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\right)$$

z-direction:

$$\rho\left(\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}\right)=$$

$$\rho g_z-\frac{\partial p}{\partial z}+\mu\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}\right)$$

$$\rho \frac{Du}{Dt} = \rho g_x - \frac{\partial P}{\partial x} + \mu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right)$$

$$\rho \frac{Dv}{Dt} = \rho g_y - \frac{\partial P}{\partial y} + \mu \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} \right)$$

$$\underbrace{\rho \frac{Dw}{Dt}}_{\text{MOMENTUM CHANGE}} = \underbrace{\rho g_z}_{\text{GRAVITY FORCE}} \underbrace{- \frac{\partial P}{\partial z}}_{\text{PRESSURE FORCE}} + \underbrace{\mu \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} \right)}_{\text{VISCOUS FORCES}}$$

NAVIER-STOKES EQUATIONS

(NEWTONIAN, INCOMPRESSIBLE FLUID)

MOMENTUM CHANGE

GRAVITY FORCE

PRESSURE FORCE

VISCOUS FORCES

"SUBSTANTIAL" DERIVATIVE

$$\frac{D(\cdot)}{Dt} = \frac{\partial(\cdot)}{\partial t} + u \frac{\partial(\cdot)}{\partial x} + v \frac{\partial(\cdot)}{\partial y} + w \frac{\partial(\cdot)}{\partial z}$$

DERIVATIVE FOLLOWING MOTION OF A FLUID ELEMENT

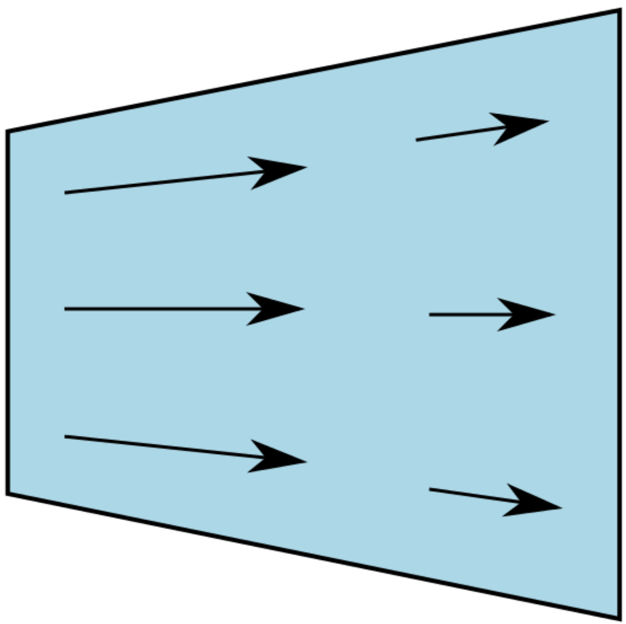
Local 'Temporal' Acceleration: velocity change with time

Convective Acceleration: velocity change with position – following the motion

Continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0$$

N-S Eqs – Convective Acceleration



Though the flow may be steady (time-independent), the fluid decelerates as it moves down the diverging duct (assuming incompressible or subsonic compressible flow), hence there is an acceleration happening over position.

N-S eqs - **Nonlinearity**

- The nonlinearity is due to **convective acceleration**
- convective acceleration is associated with the change in velocity **over position**. Hence, any convective flow, whether turbulent or not, will involve nonlinearity.

Stress Tensor τ_{ij}

τ_{ij}

i indicates the face on which the stress is acting

j indicates the direction of the stress.

$$\sigma_{ii} = \tau_{ij}$$

normal stress on face i

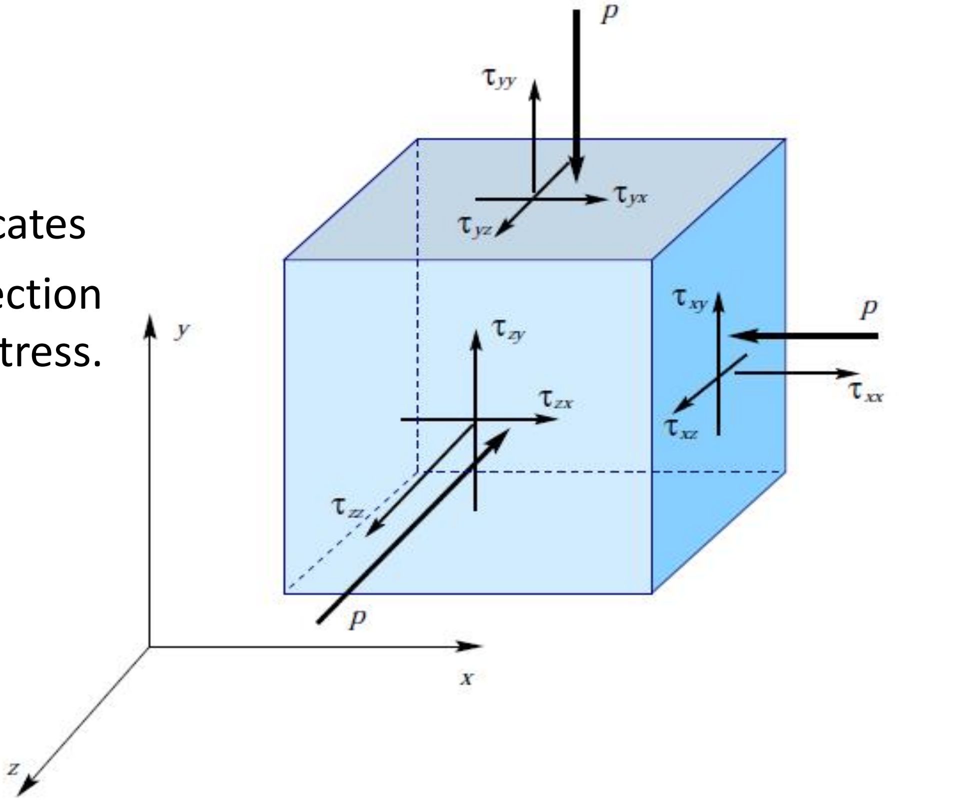


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Figure 3.9: Schematic of pressure and viscous stresses acting on a fluid element.

N-S eqs: Alternative form

- Incompressible

$$u_t + uu_x + vu_y + wu_z = -\frac{1}{\rho} p_x + \nu \Delta u + \frac{1}{\rho} F_{B,x} \,, \tag{3.43a}$$

$$v_t + uv_x + vv_y + wv_z = -\frac{1}{\rho} p_y + \nu \Delta v + \frac{1}{\rho} F_{B,y} \,, \tag{3.43b}$$

$$w_t + uw_x + vw_y + ww_z = -\frac{1}{\rho} p_z + \nu \Delta w + \frac{1}{\rho} F_{B,z} \,. \tag{3.43c}$$

Here, ν is *kinematic viscosity*, the ratio of viscosity μ to density ρ, as given earlier in Chap. 2, and Δ is the second-order partial differential operator (given here in Cartesian coordinates)

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

known as the *Laplacian* or *Laplace operator* (which is usually denoted by ∇^2 in the engineering and

$$\overbrace{\underbrace{u_t}_{\substack{local\\accel}} + \underbrace{uu_x + vu_y + wu_z}_{\substack{convective\\accel}}}^{total\ acceleration} = \overbrace{-\frac{1}{\rho}p_x}^{\substack{pressure\\forces}} + \overbrace{\nu\Delta u}^{\substack{viscous\\forces}} + \overbrace{\frac{1}{\rho}F_{B,x}}^{\substack{body\\forces}} \; .$$

N-S eqs - Viscous Forces

$$\nu \Delta u = \nu(u_{xx} + u_{yy} + u_{zz})$$

- Viscosity terms given above are associated with molecular transport (i.e., **diffusion**) of momentum.
- In general, second derivative terms in a differential equation are usually associated with **diffusion**, and in both physical and mathematical contexts this represents smoothing, or mixing process

High- and low-viscosity fluids

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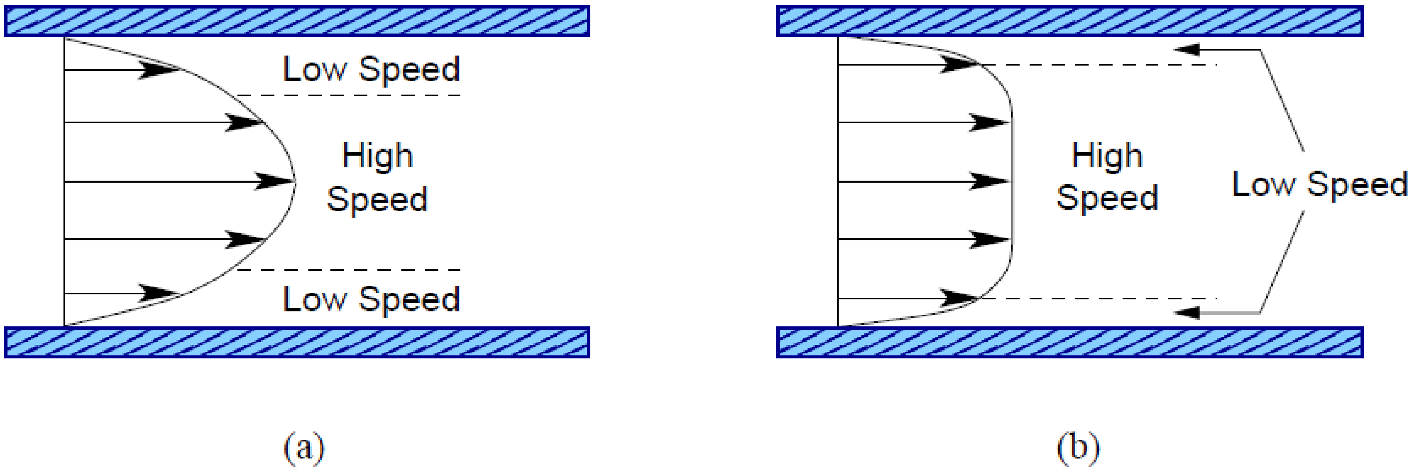


gure 3.11: Comparison of velocity profiles in duct flow for cases of (a) high viscosity, and (b) low
scosity.

Viscous Forces – Boundary Layers

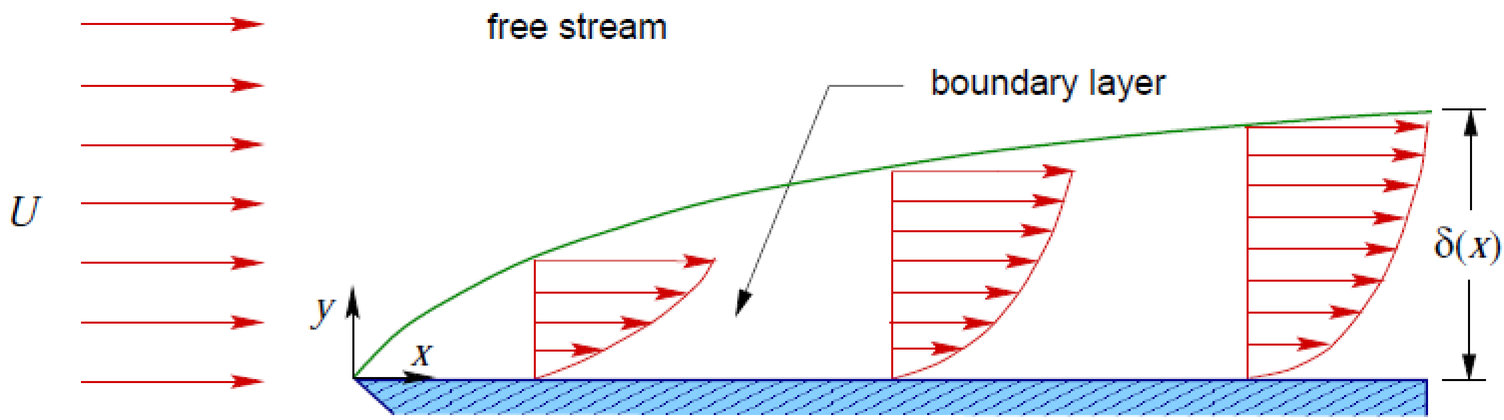
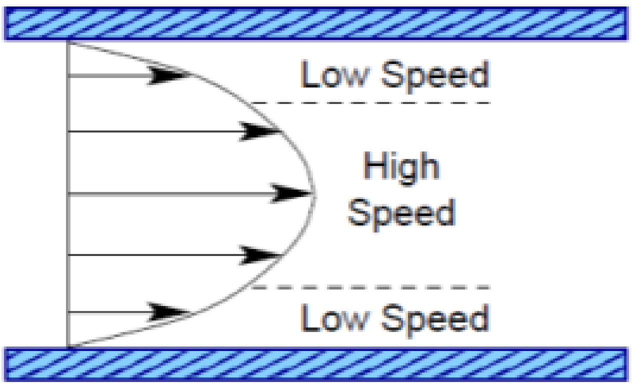


Figure 4.12: Steady, 2-D boundary-layer flow over a flat plate.

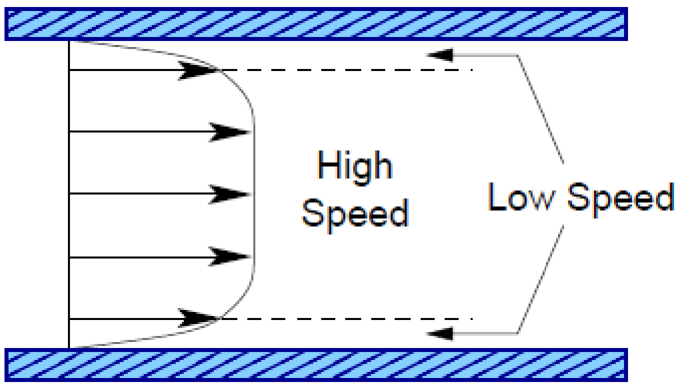
High viscosity fluids



(a)

- Velocity profile varies smoothly coming away from zero velocity at the wall, and reaching a maximum velocity in the center of the duct.
- Large viscosity: **diffusion** of viscous forces (time-rate of change of momentum) arising from high shear stress near the solid surfaces far into the flow field, thus smoothing the entire velocity profile

low-viscosity fluids



(b)

- narrow region of low-speed flow near the solid boundaries and a wider region of nearly constant-velocity flow in the central region of the duct

- speed on the centerline: is lower (for the same mass flow rate) than would be the high viscosity case.

Solution of N-S eqs

- **No** general analytical solution
  - **Millennium Prize Problems**

- Analytical solutions for few simple flow problems
- Complex flow problems:
  - Experimental investigation (cost, time) –
    - may be used to validate numerical solutions
  - Numerical solutions (magic alternative)

Millennium Prize Problems

- Seven problems in mathematics that were stated by the Clay Mathematics Institute in 2000.

- A correct solution to any of the problems results in a US $1 million prize being awarded by the institute to the discoverer(s).

- The problems are Birch and Swinnerton-Dyer conjecture, Hodge conjecture, Navier–Stokes existence and smoothness, P versus NP problem, Poincaré conjecture, Riemann hypothesis, and Yang–Mills existence and mass gap.

- At present, the only Millennium Prize problem to have been solved is the Poincaré conjecture, which was solved by the Russian mathematician Grigori Perelman in 2003.

Solution of N-S eqs

- Numerical solutions (magic alternative)
- http://www.engr.uky.edu/~acfd/lecturenotes1.html

The Strategy of CFD

Replace the continuous problem domain with a discrete domain using a grid.

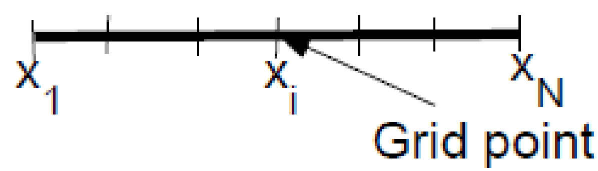
Continuous Domain

$0 \leq x \leq 1$

Coupled PDEs + boundary conditions in continuous variables

Discrete Domain

$x = x_1, x_2, \ldots, x_N$



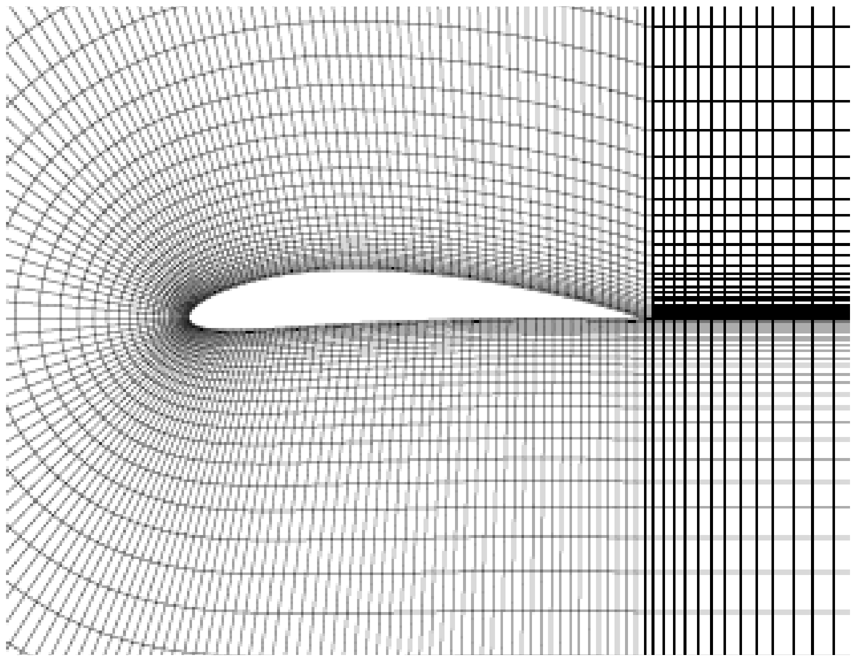
Coupled algebraic eqs. in discrete variables

CFD - discrete system

- The governing partial differential equations and boundary conditions are defined in terms of the continuous variables p, u, v, w etc

- The discrete system is a large set of **coupled algebraic equations** in the discrete variables.

CFD: Grid - Mesh



- **Parallel flow[edit]**
- Assume steady, parallel, one dimensional, non-convective pressure-driven flow between parallel plates, the resulting scaled (dimensionless) boundary value problem is:
- d 2 u d y 2 = − 1 ; u (0) = u (1) = 0. {\displaystyle {\frac {d^{2}u}{dy^{2}}}=-1;\quad u(0)=u(1)=0.} The boundary condition is the no slip condition. This problem is easily solved for the flow field:
- u (y) = y − y 2 2 . {\displaystyle u(y)={\frac {y-y^{2}}{2}}.} From this point onward more quantities of interest can be easily obtained, such as viscous drag force or net flow rate.

GENERAL SOLUTION PROCEDURE **Navier-Stokes Eqs**

① CHOOSE A COORDINATE SYSTEM

ASK YOURSELF: A) WHAT DIRECTION IS THE FLOW?

B) IN WHAT DIRECTION DOES THE VELOCITY CHANGE?

② DETERMINE THE DRIVING FORCE FOR FLOW (PRESSURE, SHEAR, GRAVITY)

③ WRITE THE BOUNDARY CONDITIONS (BCs)

④ GUESS THE FORM OF THE SOLUTION → WHAT SHOULD IT LOOK LIKE?

⑤ SIMPLIFY THE CONSERVATION EQUATIONS

⑥ SOLVE THE RESULTING DIFFERENTIAL EQUATION

Couette flow

flow between to infinite parallel plates spaced a distance h apart in the y direction

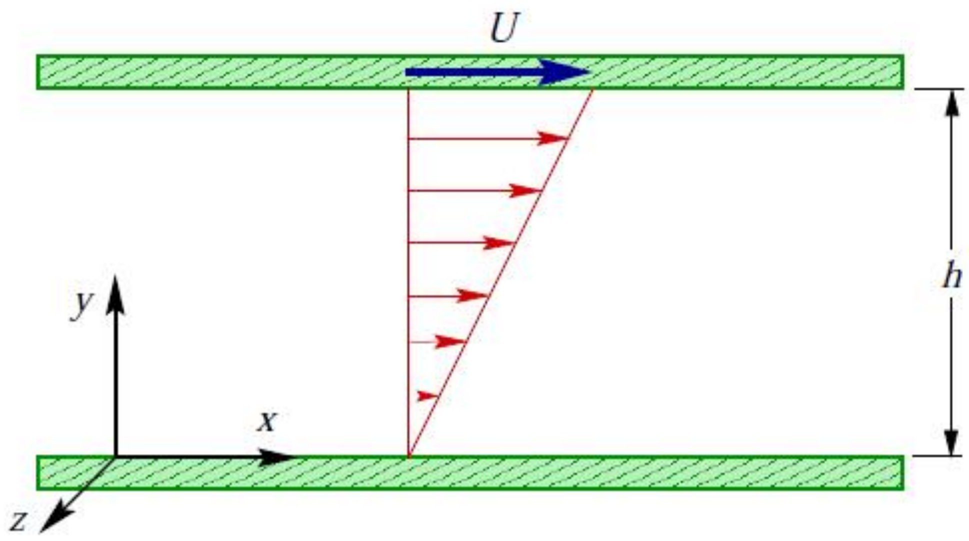


Figure 4.9: Couette flow velocity profile.

The Hagen–Poiseuille solution

- steady, incompressible, axisymmetric, fully-developed, laminar flow, P1 > P2

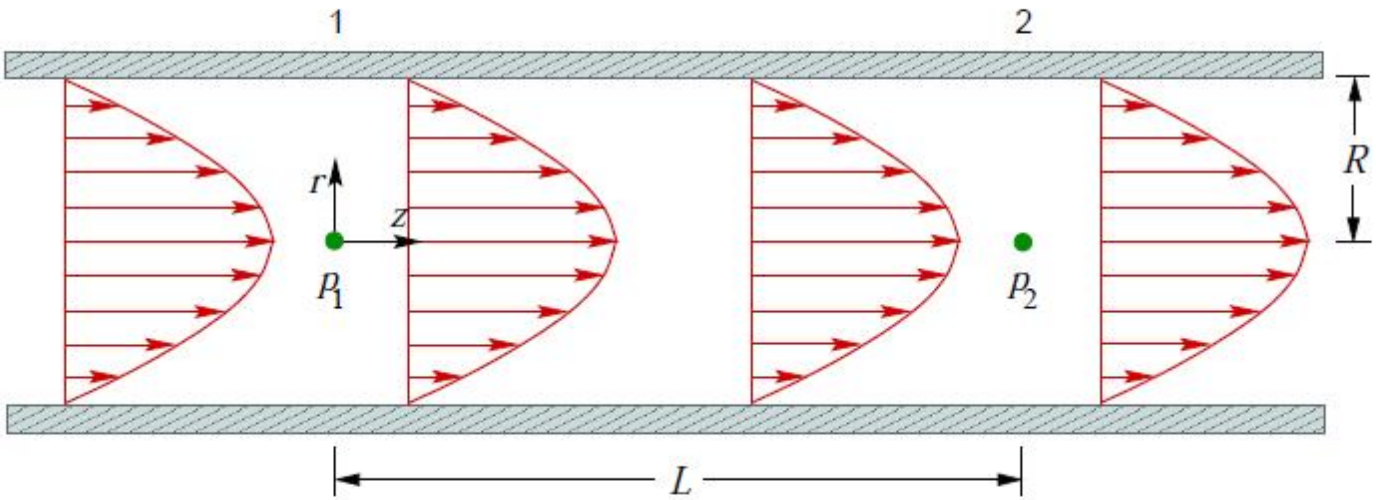


Figure 4.13: Steady, fully-developed pipe flow.

Ex: Possible flow field

- Use the continuity eq to check the possibility of the following steady incompressible flow with velocity field

$$u(x, y, z) = 2x + y + z \ , \ v(x, y, z) = -y \ , \ w(x, y, z) = -z \ .$$

Ex: Local & convective accelerations -1

Find the local and convective accelerations assuming a velocity field with the 3 components:

u = x + y + z + t , v = x2y3zt , w = exp(xyzt)

Solution

$$\frac{DU}{Dt} = \frac{\partial U}{\partial t} + U \cdot \nabla U \, ,$$

Ex: Local & convective accelerations -2

$$a_x \equiv \frac{Du}{Dt} = \frac{\partial u}{\partial t} + U \cdot \nabla u\,,$$

$$a_x = u_t + uu_x + vu_y + wu_z\,,$$

$$\begin{aligned} a_y &= v_t + uv_x + vv_y + wv_z\,, \\ a_z &= w_t + uw_x + vw_y + ww_z\,. \end{aligned}$$

Ex: Local & convective accelerations -3

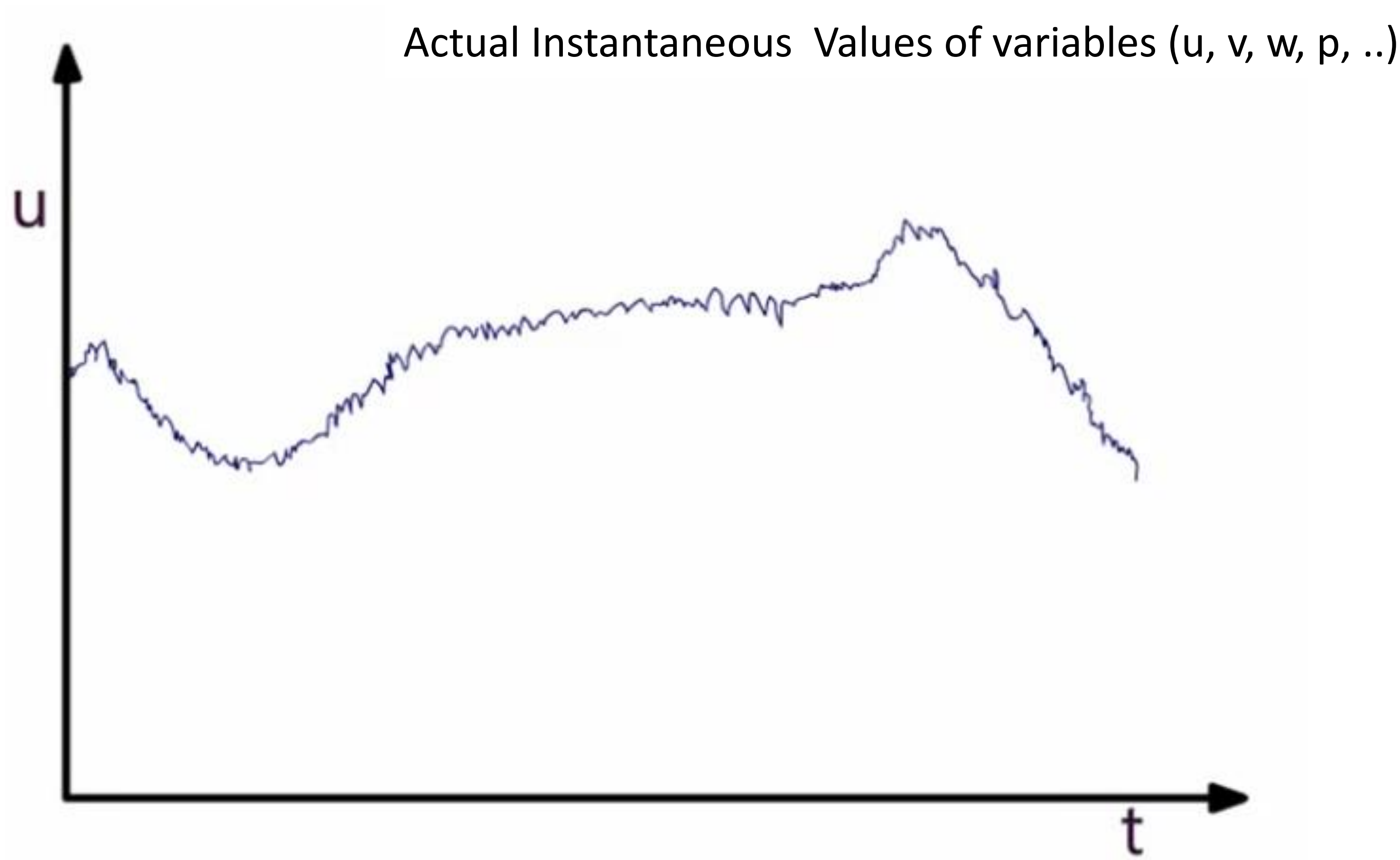
$$u_t = 1\,, \qquad u_x = 1\,, \qquad u_y = 1\,, \qquad u_z = 1\,,$$

$$v_t = x^2y^3z\,, \qquad v_x = 2xy^3zt\,, \qquad v_y = 3x^2y^2zt\,, \qquad v_z = x^2y^3t\,,$$

$$w_t = xyz\exp(xyzt)\,, \qquad w_x = yzt\exp(xyzt)\,,$$

$$w_y = xzt\exp(xyzt)\,, \qquad w_z = xyt\exp(xyzt)\,.$$

Actual Instantaneous values



Averaging process of N-S eqs

Average velocity

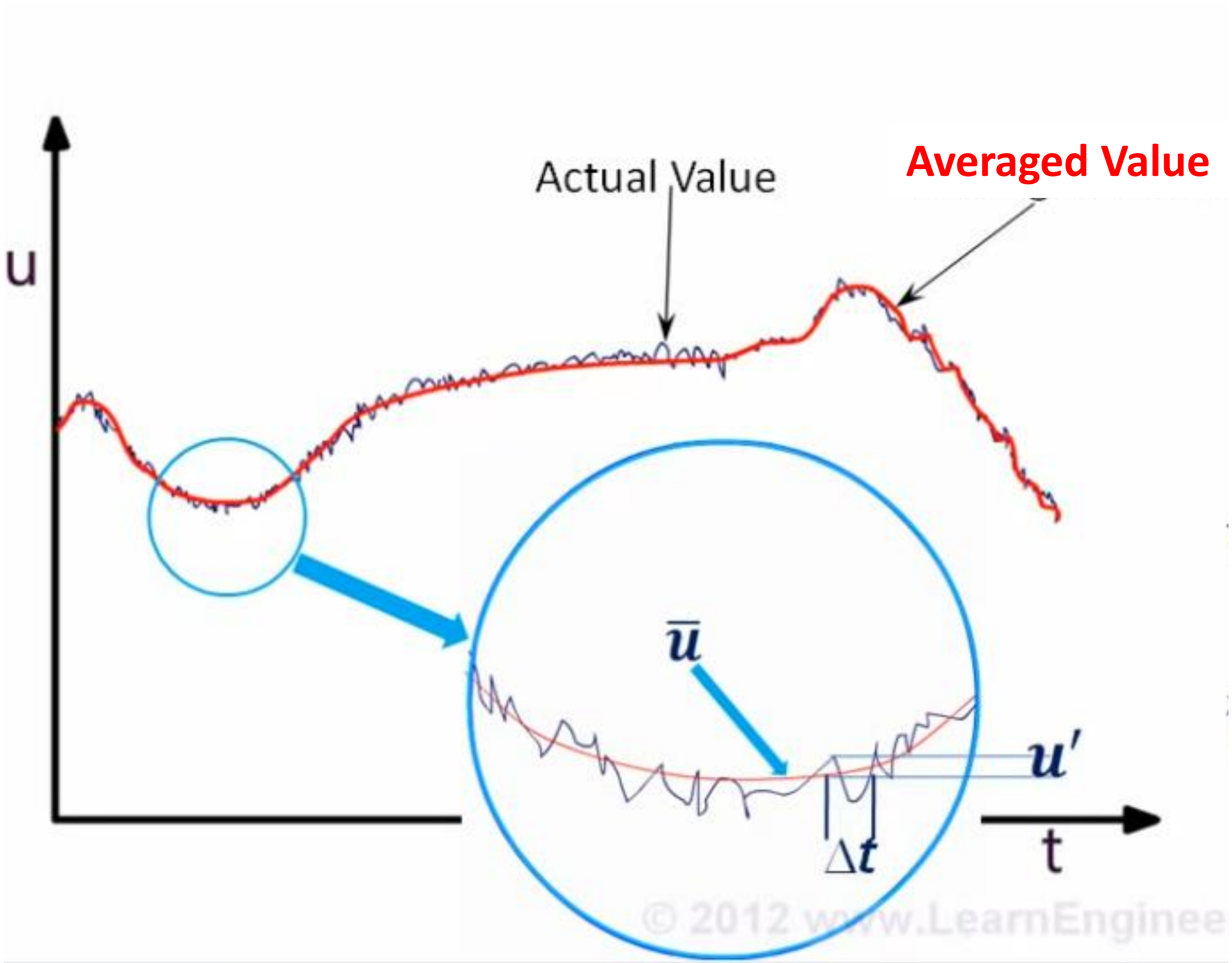
Actual velocity

Fluctuating velocity

$$\boldsymbol{u} = \overline{\boldsymbol{u}} + \boldsymbol{u'}$$

$$\overline{\boldsymbol{u}} = \frac{1}{\Delta t}\int_{t}^{t+\Delta t} \boldsymbol{u}\, dt$$

Averaging process of N-S eqs



RANS: Reynolds Average Navier Stokes

RANS

Reynolds Stress

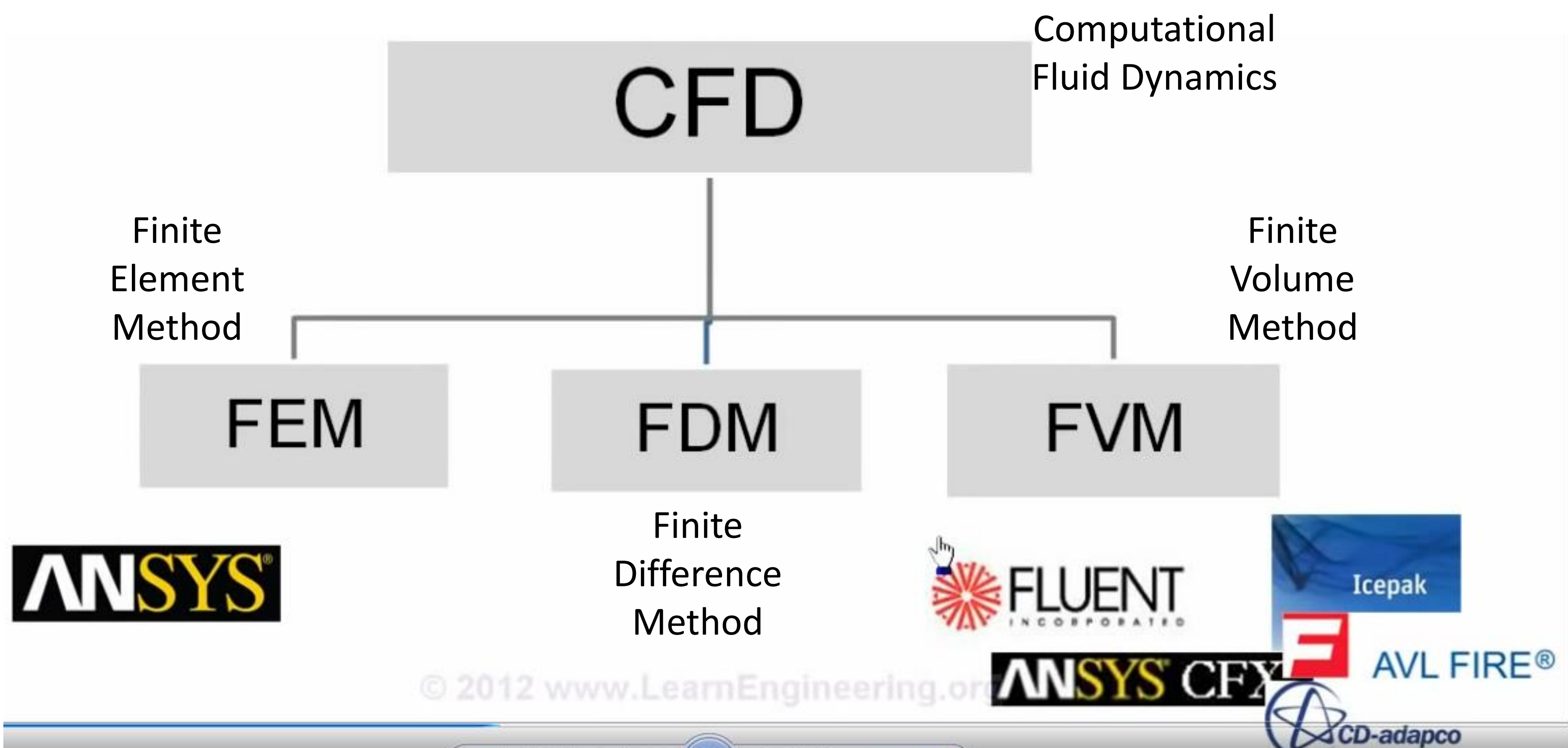
$$\rho \bar{u}_j \frac{\partial \bar{u}_i}{\partial x_j} = \rho \bar{f}_i + \frac{\partial}{\partial x_j}\left[-\bar{p}\delta_{ij} + \mu\left(\frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial \bar{u}_j}{\partial x_i} \right) - \overline{\rho u_i' u_j'} \right]$$

$$i, j = 1,2,3$$

$$\delta_{ij} = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$$

Solving RANS

- RANS: Reynolds Average Navier Stokes



Numerical Solutions

N-S eqs: Nonlinear **PDEs** ‘Partial Differential Equations ’ + BCs + ICs

Split Domain into grid / Mesh

Discretization at mesh nodes:

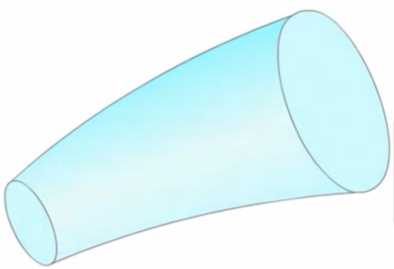
Finite Difference – Finite Element – Finite Volume

Set of simultaneous linear **algebraic** eqs

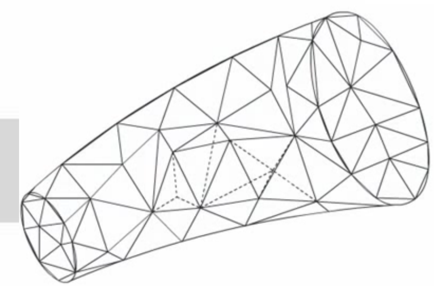
CFD FVM Steps

Integral NS to RANS

Turbulence Model ?



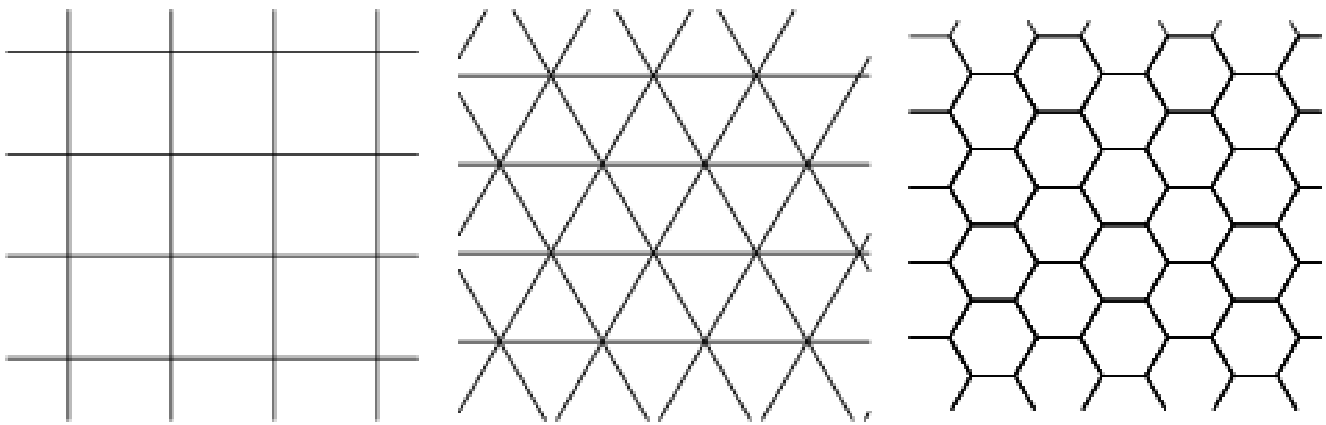
Split the CV into small cells



Numerical Integration of RANS throughout the cells

Solve numerical equations for u,v,w and p with help of B.Cs

Grid / Mesh in 2-D

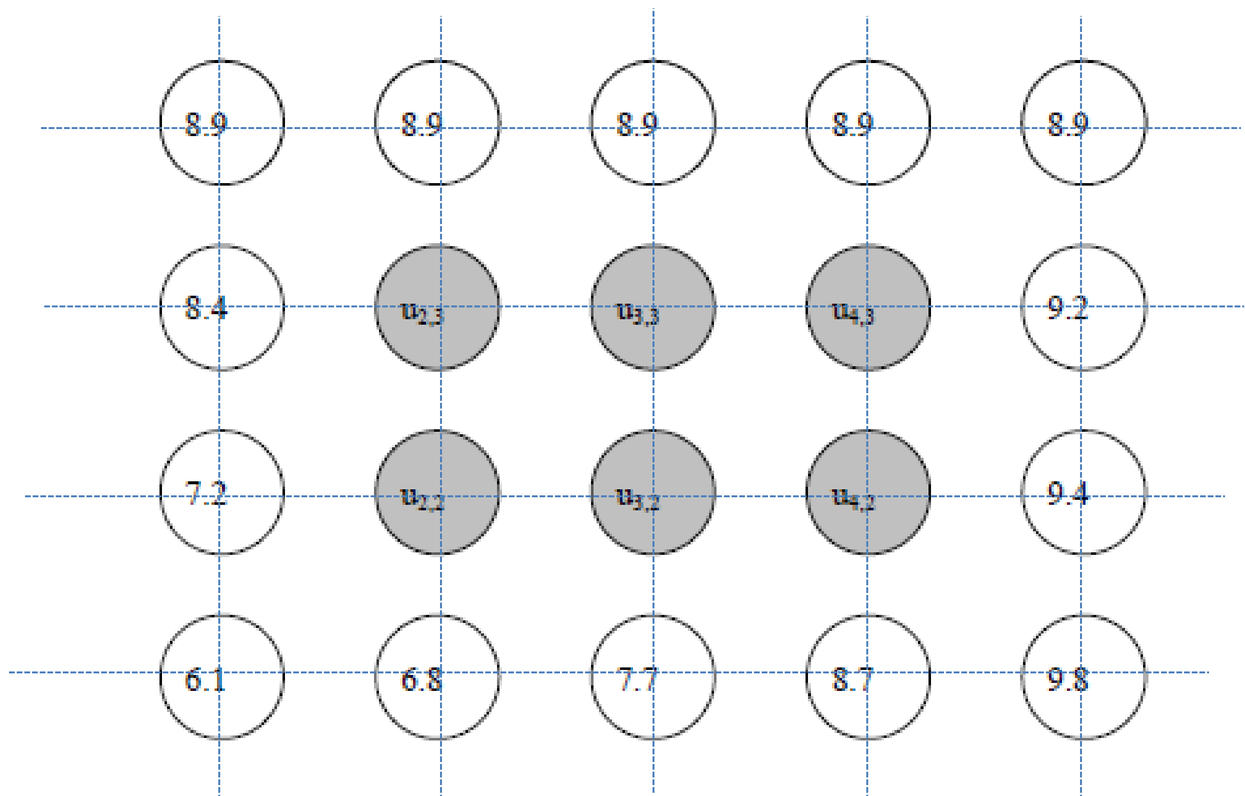


Discretization

partial derivative	finite difference approximation	type	order
$\frac{\partial U}{\partial x} = U_x$	$\frac{U_{i+1}^n - U_i^n}{\Delta x}$	forward	first in x
$\frac{\partial U}{\partial x} = U_x$	$\frac{U_i^n - U_{i-1}^n}{\Delta x}$	backward	first in x
$\frac{\partial U}{\partial x} = U_x$	$\frac{U_{i+1}^n - U_{i-1}^n}{2\Delta x}$	central	second in x
$\frac{\partial^2 U}{\partial x^2} = U_{xx}$	$\frac{U_{i+1}^n - 2U_i^n + U_{i-1}^n}{\Delta x^2}$	symmetric	second in x
$\frac{\partial U}{\partial t} = U_t$	$\frac{U_i^{n+1} - U_i^n}{\Delta t}$	forward	first in t
$\frac{\partial U}{\partial t} = U_t$	$\frac{U_i^n - U_i^{n-1}}{\Delta t}$	backward	first in t
$\frac{\partial U}{\partial t} = U_t$	$\frac{U_i^{n+1} - U_i^{n-1}}{2\Delta t}$	central	second in t
$\frac{\partial^2 U}{\partial t^2} = U_{tt}$	$\frac{U_i^{n+1} - 2U_i^n + U_i^{n-1}}{\Delta t^2}$	symmetric	second in t

Interior and Boundary Conditions

- Interior: Shaded
- Boundary: White



Finite Difference Method for Solving Elliptic PDE's

– Based on Boundary Conditions (BCs) and finite difference approximation to formulate system of equations

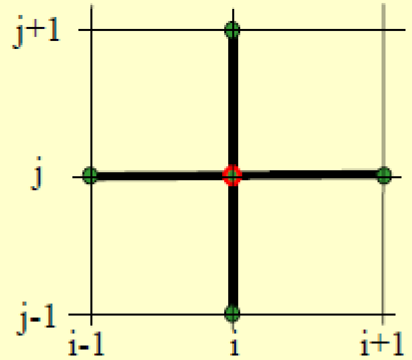
– Use Gauss-Seidel to solve the system

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \begin{cases} 0 & \textbf{Laplace Eq.} \\ -D\left(x, y, u, \dfrac{\partial u}{\partial x}, \dfrac{\partial u}{\partial y}\right) & \textbf{Poisson Eq.} \end{cases}$$

1. Discretize domain into grid of evenly spaced points
2. For nodes where u is unknown:
w/ $\Delta x = \Delta y = h$, substitute into main equation
3. Using Boundary Conditions, write, n*m equations for
u(xi=1:m, yj=1:n) or n*m unknowns.
4. Solve this banded system

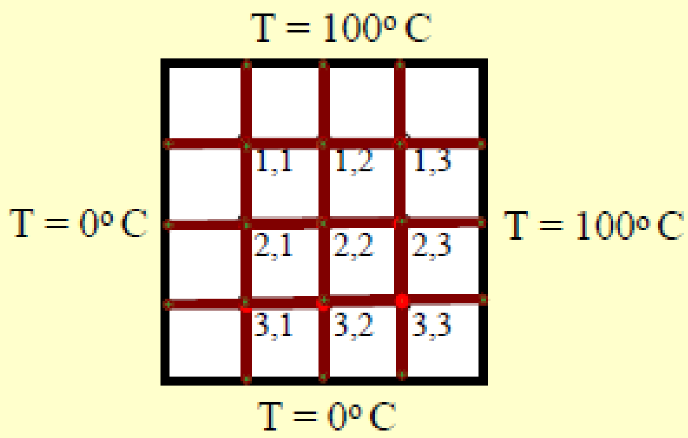
Laplace Eq

The Laplace molecule



If $\Delta x = \Delta y$ then $$T_{i+1,j} + T_{i-1,j} + T_{i,j+1} + T_{i,j-1} - 4T_{i,j} = 0$$

Laplace Eq



The temperature distribution can be estimated by discretizing the Laplace equation at 9 points and solving the system of linear equations.

$$
\begin{array}{c}
\begin{array}{ccccccccc} T_{11} & T_{12} & T_{13} & T_{21} & T_{22} & T_{23} & T_{31} & T_{32} & T_{33} \end{array} \\
\left[\begin{array}{ccccccccc}
-4 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & -4 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & -4 & 0 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & -4 & 1 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 1 & -4 & 1 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 1 & -4 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & -4 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 1 & -4 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & -4
\end{array}\right]
\end{array}
\begin{Bmatrix} T_{11} \\ T_{12} \\ T_{13} \\ T_{21} \\ T_{22} \\ T_{23} \\ T_{31} \\ T_{32} \\ T_{33} \end{Bmatrix}
=
\begin{Bmatrix} -100 \\ -100 \\ -200 \\ 0 \\ 0 \\ -100 \\ 0 \\ 0 \\ -100 \end{Bmatrix}
$$

Excel